Conical Transport in Complex Honeycomb Lattices with $\mathcal{PT}$ Symmetry

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Beam dynamics in complex honeycomb photonic lattices with local $\mathcal{PT}$ symmetries imposed by a balanced arrangement of gain or loss is investigated. A new type of conical diffraction that is a consequence of the spontaneous $\mathcal{PT}$-symmetry breaking phase transition can be enforced by controlling the gain/loss parameter $\gamma$. The evolving cone is brighter and propagates along the lattice with a transverse speed which is proportional to $\sqrt{\gamma}$.

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Introduction – The phenomenon of conical refraction i.e the spreading into a hollow cone of an unpolarized light beam entering a biaxial crystal along its optic axis, is fundamental in crystal optics and in mathematical physics [1–4]. It was first predicted by Hamilton [3] and observed by Lloyd [4] soon afterwards, while it has been intensively studied, both theoretically and experimentally, since then [1–8]. The physical origin of the phenomenon is associated with the existence of the so-called diabolical points (DP) which emerge along the axis of intersection of the two shells associated with the wave surface. Around a DP the energy dispersion relation is linear while the direction of the group velocity is not uniquely defined. Recently conical diffraction (CD) was observed in 2D photonic honeycomb lattices [5] which shares many common features (like the existence of DPs) with the band structure of graphene in solid state physics.

In graphene, the electrons around the DPs of the band structure behave as “massless relativistic fermions”, thus resulting in extremely high electron mobility. Both photonic and electronic graphene structures allow us to test experimentally various predictions of relativistic quantum mechanics like, the Klein paradox [9], the existence of exotic particles like optical tachyons [10] etc.

While DPs are spectral singularities associated with Hermitian systems, for pseudo-Hermitian Hamiltonians, like those used for the theoretical description of non-Hermitian optics, a topologically different singularity may appear: an exceptional point (EP), where not only the eigenvalues but also the associated eigenstates coalesce. Pseudo-Hermitian optics is a rapidly developed field which aims via a judicious design that involves the combination of delicately balanced amplification and absorption regions together with the modulation of the index of refraction, to achieve new classes of synthetic meta-materials that can give rise to altogether new physical behavior and novel functionality [11, 12]. The idea can be carried out via index-guided geometries with special antilinear symmetries. Adopting a Schrödinger language (applicable in the paraxial approximation), the effective Hamiltonian that describes the optical beam evolution is non-Hermitian and commutes with the combined parity ($P$) and time ($T$) operator [13–15]. In optics, $\mathcal{PT}$ symmetry demands that the complex refractive index obeys the condition $n(r) = n^*(-r)$. It can be shown that for such structures, a real propagation constant (eigenenergies in the Hamiltonian language) exists for some range (the so-called exact phase) of the gain/loss coefficient. For larger values of this coefficient the system undergoes a spontaneous symmetry breaking, corresponding to a transition from real to complex spectra (the so-called broken phase). The phase transition point, shows all the characteristics of an exceptional point (EP) singularity.

$\mathcal{PT}$-synthetic materials can exhibit several intriguing features. These include among others, power oscillations and non-reciprocity of light propagation [11, 16, 18, 19], non-reciprocal Bloch oscillations [20], optical tachyons in graphene [10] and unidirectional invisibility [21]. In the nonlinear domain, such pseudo-Hermitian non-reciprocal effects can be used to realize a new generation of on-chip isolators and circulators [22]. Other results within the framework of $\mathcal{PT}$-optics include the realization of coherent perfect laser absorber [23] and nonlinear switching structures [24]. Despite the wealth of results on transport properties of $\mathcal{PT}$-symmetric 1D optical structures, the properties of high dimensional $\mathcal{PT}$ optical lattices, (with the exception of few recent studies [11, 25]), has remained so far essentially unexplored.

In this Letter we study beam propagation in non-Hermitian 2D optical honeycomb lattices with $\mathcal{PT}$-symmetry and investigate the possibility of abnormal diffraction. We find a new type of conical diffrac-
tion which is associated with the spontaneously $\mathcal{PT}$-symmetry breaking phase transition point. Despite the fact that at the EP the Hilbert space collapses the emerging cone is brighter and propagates with a transverse velocity that is controlled by the gain/loss parameter $\gamma$.

Model – We consider a two-dimensional (2D) honeycomb photonic lattice of coupled optical waveguides. Each of the waveguides can support only one mode, while light is transferred from waveguide to waveguide through optical tunneling. A schematic of the set-up is shown in Fig. 1. The lattice consist of two types of waveguides: type (A) made from lossy material (green) whereas type (B) exhibits the equal amount of gain (red). Their arrangement in space is such that they form coupled (A-B) dimers with inter/intra-dimer couplings $t_a$ and $t$ respectively. Such structure, apart from a global $\mathcal{PT}$ symmetry, it respects also another anti-linear symmetry (in Ref. [26] we coined this $\mathcal{P}_{\eta}$-symmetry) which is related with the local $\mathcal{PT}$-symmetry of each individual dimer.

Without loss of generality we measure everything in units of the inter-dimer coupling $t_a = 1$. In the tight binding description, the diffraction dynamics of the mode electric field amplitude $\Psi_{n,m}$ in their Fourier representation i.e. $a_{n,m}(z) = \frac{1}{2\pi} \int \pi d k_x d k_y \tilde{a}_{n,m}(k) \exp(i [m k_x + n k_y])$ (and similarly for $b_{n,m}$). Substitution to Eq. (1) leads to

$$
\frac{id}{dz} \begin{pmatrix} a_{k}(z) \\ b_{k}(z) \end{pmatrix} = \begin{pmatrix} H_k & D(k) \\ (D(k)^*) & H_k \end{pmatrix} \begin{pmatrix} a_{k}(z) \\ b_{k}(z) \end{pmatrix} = \begin{pmatrix} -i \gamma & D(k) \\ D(k)^* & i \gamma \end{pmatrix} \begin{pmatrix} a_{k}(z) \\ b_{k}(z) \end{pmatrix}
$$

(2)

where $k \equiv (k_x, k_y)$ and $D(k) = -(t + 2e^{-i k z} \cos k_y)$.

Eigenmodes Analysis – Substituting in Eq. (2) the stationary form $(\tilde{a}_{n,m}, \tilde{b}_{n,m})^T = \exp(-i \mathcal{E} z) (A, B)^T$ we get

$$
\mathcal{E} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} H_k & D(k) \\ (D(k)^*) & H_k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
$$

(3)

The spectrum is obtained by requesting a non-trivial solution i.e. $(A, B) \neq 0$. The corresponding dispersion relation [10] has the form

$$
\mathcal{E}_\pm = \pm \sqrt{|D(k)|^2 - \gamma^2}
$$

(4)

For $\gamma = 0$ the dispersion relation is $\mathcal{E} = \pm |D(k)|$ and we have two bands of width $t + 2$. There are three pairs of diabolic points, $k_{0,\pm}^\pm = (\pm \pi, \pm \pi)$ and $k_{0,\pm}^\pm = (0, \pm \pi \pm \pi)$. Expansion of $D(k)$ up to the first order around the DP leads to a linear dispersion relation $\mathcal{E}_\pm \approx \pm (t^2 \eta^2_t + (4 - t^2) \eta^2_d \mp \pi)$ where $\eta_{x,y} = k_{x,y} - (0, 0)$.

The standard passive ($\gamma = 0$) honeycomb lattice (zero strain) corresponds to $t = 1$. In this case, there are three pairs of DPs at $k_{0,\pm}^\pm = (0, \pm \pi)$ and $k_{0,\pm}^\pm = (0, \pm \pi)$. For $1 < t < 2$ the two pairs of DPs at $k_{0,\pm}^\pm$ start moving toward each other while the pair at $k_{0,\pm}^\pm$ moves away from one another. At $t = 2$ a “degeneracy” occurs i.e. $k_{0,\pm}^\pm = (\pm \pi, 0)$. At the same time the dispersion relation $\mathcal{E}$ around $k_{0,\pm}^\pm$ and $k_{0,\pm}^\pm$ is linear only in the $k_x$ direction (and quadratic in the $k_y$). For $t > 2$ the two energy surfaces move away from each other and a gap between them is created. Therefore for $t > 2$ the DPs disappeared for $\gamma = 0$, and the conical diffraction is destroyed [6, 27].

On the other hand, the size of the gap between the two bands (in case of strain $t > 2$) can be controlled by manipulating the gain/loss parameter $\gamma$. In this case, the system of Eq. (1) becomes non-Hermitian. Nevertheless, because the Hamiltonian commutes with the $\mathcal{PT}$-operator there is a $\gamma$ domain for which the energies are real. It turns out from Eq. (4) that the line $\mathcal{E}_\pm = \pm 2$ defines the phase transition from exact to broken $\mathcal{PT}$-symmetry [10]. The mechanism for this breaking is level crossing between levels (corresponding to $(k_x, k_y) = (0, \pm \pi)$ or $(k_x, k_y) = (\pm \pi, 0)$) belonging to different bands [26]: it follows from Eq. (4), that when $\gamma = \gamma_\mathcal{PT}$, the gap disappears and the two (real) levels at the “inner” band-edges become degenerate. Evaluation of $D(k_x, k_y)$ to second order in $(\eta_x, \eta_y)$ around the degeneracy points, leads to

$$
D(\eta_x, \eta_y) \approx -(\gamma_\mathcal{PT} + 2\eta_x + \eta^2); \quad \eta^2 = \eta_x^2 + \eta_y^2
$$

(5)
thus resulting in the following dispersion relation

\[ \mathcal{E} = \pm \sqrt{2\gamma_\tau T \eta^2 + (2\gamma_\tau T + 4)\eta^2} \approx \pm \sqrt{2\gamma_\tau T \eta}. \]  

The last approximation applies in the case of large \( \gamma_\tau T \)-values. This comment will become important in the analysis of optical beam propagation that follows.

The eigenvectors of a non-Hermitian system are bi-orthogonal, and therefore do not respect the standard (Euclidian) orthogonal-normalization condition. Let \( \langle L_n | \rangle \) and \( |R_n) \) denote the left and right eigenvectors of the non-hermitian Hamiltonian \( \mathcal{H} \) corresponding to the eigenvalue \( \mathcal{E}_n \), i.e. \( \langle L_n | \mathcal{H} = \langle L_n | \mathcal{E}_n \rangle \) and \( \mathcal{H}|R_n) = \mathcal{E}_n |R_n) \). The vectors can be normalized to satisfy \( \langle L_n | \mathcal{H} \rangle \delta_{nm} \) while \( \sum_n \langle L_n | \mathcal{E}_n \rangle = \delta_{nm} \) (\( N \) is the dimension of the Hilbert space). An observable that measures the non-orthogonality of the system is the so-called Petermann factor, defined as \( K_{nm} = \langle L_n | \mathcal{H} | R_m) \rangle \) [28]. We have studied the mean (diagonal) Petermann factor \( \overline{K} \equiv \frac{1}{N} \sum_{n=1}^{N} K_{nn} \) which take the value 1 if the eigenfunctions of the system are orthogonal while is larger than one in the opposite case. At the EP, a pair of eigenvectors associated with the corresponding degenerate eigenvalues coalesce, leading to a "collapse" of the Hilbert space. At this point, the Petermann factor diverges as \( \overline{K} \sim 1/|\gamma_\tau T| \) [18, 29]. This indicates strong correlations between the spectrum and the eigenvectors which can affect drastically the dynamics as we will see later. In Figs. 2d-f we have calculated \( \overline{K} \) for different \( t, \gamma_\tau \) values and various system sizes and found that as \( N \) increases the divergence is approaching the line \( \gamma_\tau T = t - 2 \).

**Dynamics** — We first study wave propagation in the honeycomb lattice numerically (Fig. 3), by launching a beam with the structure of a Bloch mode associated with the EP, multiplied by a Gaussian envelope. The Bloch modes at the tip can be constructed from pairs of plane waves with \( k \) vectors of opposite pairs of exceptional points. Thus, interfering two plane-waves at angles associated with opposite EP yields the phase structure of the modes from these points. Multiplying these waves by an envelope yields a superposition of Bloch modes in a region around these points. Figure 3 shows an example of the propagation of a beam constructed to excite a Gaussian superposition of Bloch modes around an EP. The input beam has a bell-shape structure, which, after some distance, transforms into the ring-like characteristic of conical diffraction [27]. From there on, the ring is propagating in the lattice by keeping its width constant while its radius is increasing linearly with distance. The invariance of the ring thickness and structure manifests a (quasi-) linear dispersion relation above and below the EP (see Fig. 1); hence, the diffraction coefficient for wave packets constructed from Bloch modes in that region is zero (infinite effective mass). This is especially interesting because the ring itself is a manifestation of the dispersion properties at the EP itself, where the diffraction coefficient is infinite (zero-effective mass). As a result, the ring forms a light cone in the lattice. The appearance of CD in the case of \( PT \)-lattices where the eigenvectors are non-orthogonal and coalesce at the EP singularity, provides a clear indication that the phenomenon is insensitive to the eigenmode structure and it depends only on the properties of the dispersion relation.

The \( PT \)-symmetric conical diffraction shows some unique characteristics with respect to the CD found in the case of beam propagation around DPs for passive honeycomb lattices (i.e. \( \gamma = 0 \)). A profound difference is associated with the fact that now, the transverse speed of the cone is increased and in fact it can be controlled by the magnitude of the gain/loss parameter at the symmetry breaking point \( \gamma_\tau T \). This is shown in Fig. 3 where we compare the spreading of a CD for two different \( \gamma_\tau T \)-values. Detail numerical analysis indicate that the transverse velocity is proportional to \( \sqrt{\gamma_\tau T} \), thus providing a clear indication that the phenomenon is insensitive to the eigenmode structure and it depends only on the properties of the dispersion relation.

It is possible to gain valuable insight into the features of \( PT \)-conical diffraction by considering the field evolution in the momentum space. We consider for simplicity, an initial distribution \( \langle \hat{a}_k(0), \hat{b}_k(0) \rangle \) that is symmetric around the EP while it decays exponentially away from it. Specifically we assume \( \langle \hat{a}_k(0), \hat{b}_k(0) \rangle \equiv e^{-\theta_k^2} \) where \( \theta_k \) is given by Eq. (2). After a straightforward algebra and using the fact that \( \theta_k^2 = \mathcal{E}^2 \times 1 \), (where \( 1 \) is the unity matrix) we get

\[ U = \cos(z|\mathcal{E}|)1 - i(\sin(z|\mathcal{E}|)/|\mathcal{E}|)\mathbf{H}_k \]
Equation (7) is the starting point of our analysis. Substituting Eqs. (2, 5, 6), we find that the evolving amplitude of the field \((a_n,m, b_n,m)\)
is
\[
a_n,m(z) \approx \sum_{l=1,2} (-1)^l \left[ (z - i\gamma PT \phi(n,m,z,g)) + g \right]^{\frac{1}{3}}
\]
\[
b_n,m(z) \approx \sum_{l=1,2} (-1)^{l+1} \left( i \gamma PT \phi(n,m,z,g) + \frac{n}{\sqrt{2\gamma PT}} \right)^{\frac{1}{3}}
\]
where \(\phi(n,m,z,g) = \left[ g + (-1)^l z^2 \right]^{\frac{1}{3}} + n^2/(2\gamma PT) + 4 + m^2/(2\gamma PT)\). Although our simplified calculations are not able to capture all the features of the evolving cone, the above expression encompasses the main characteristics of the conical diffraction that we have observed in our numerical simulations. At \(z = 0\), Eq. (8) resembles a Lorentzian, which slowly transforms into a ring of light, whose radius expands linearly with \(z\) and velocity \(\sqrt{2\gamma PT}\), while its thickness remains unchanged. At the same time the field intensity on the ring in the case of \(\mathcal{PT}\)-symmetric lattices is brighter than the one corresponding to passive honeycomb lattices (i.e. \(1/z^2\) vs \(1/z^4\) behavior respectively).

Conclusions – In conclusion, we studied waves in \(\mathcal{PT}\)-honeycomb photonic lattices, demonstrating the existence of conical diffraction arising solely from the presence of a spontaneous \(\mathcal{PT}\)-symmetry breaking phase transition point. In spite the fact that the eigenvectors are non-orthogonal and there is a collapse of the Hilbert space at the EP, the emerging cone, is brighter and moves faster than the corresponding one of the passive structure. These ideas raise several intriguing questions. For example, how does nonlinearity affect \(\mathcal{PT}\)-symmetric conical diffraction? What is the effect of disorder [30]? Is this behavior generic for any system at the spontaneously \(\mathcal{PT}\)-symmetry breaking point? These intriguing questions are universal, and relate to any field in which waves can propagate in a periodic potential.

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[27] Here we use the term "conical" diffraction in a rather loose fashion. Specifically, when the lattice is deformed i.e. \(t \neq 1\), the CD pattern becomes elliptic [6].